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ON STABILITY IN THE SADDLE-POINT SENSE

David Levhari and Nissan Liviatan

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massachusetts
institute of
technology
50 memorial drive
cambridge, mass. 02139

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Introduction

In the following paper we discuss some properties of the optimal path in the multiple capital goods case. We denote by k the n-dimensional vector of per capita capital and by $\dot{k}$ the rates of change per unit of time. We assume that the aim is to maximize $\int_0^\infty e^{-\delta t} V(k,\dot{k})\,dt$ where V gives the maximum utility obtained from consumption with given k and $\dot{k}$, and δ is the rate of discount.[1)] This is the case discussed in numerous papers and here we focus our attention on the behavior of the Euler differential equations around the steady state.

Samuelson [5] has proved that for the case $\delta=0$ we have, at an optimal steady state, a saddle-point with the characteristic roots coming in pairs of λ and $-\lambda$. For the analogues discrete model he has proved [6,7] that the roots come in reciprocals λ and $\frac{1}{\lambda}$. Thus, in both cases the behavior of the path around the steady state is that of a saddle-point. We refer to this situation as "stability in the saddle-point sense".

Kurz [1] has generalized this Samuelson-Poincare theorem by considering the case of $\delta > 0$, that is the case of positive discount rate. Kurz proves for this case that it is impossible for the optimal path to be a damped stable one; either we have a saddle-point around the steady state or complete instability. Thus an optimal steady

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1) See similar formulation in Samuelson and Solow [4].

state may be stable or unstable in the saddle-point sense.[2]
Kurz's presentation is different from that of Samuelson since his treatment is based on a generalized system which includes the shadow prices as well.

In this paper we give an alternative (and considerably simpler) proof to the Kurz theorem within the framework of classical calculus of variations, as Samuelson is using, without introducing dual variables. We prove for $\delta > 0$ that if λ is a characteristic root then $-\lambda+\delta$ is also a characteristic root.

Another problem which we take up in this paper concerns the possibility of having purely imaginary roots λ and $-\lambda$ for the case $\delta=0$. If this were possible then it would imply lack of saddle-point stability for an optimal steady state with $\delta=0$, which seems very strange. Indeed, Samuelson's writings seem to imply [5,7] that for the case $\delta=0$ and concave production function and utility function (that is strictly concave V) it is impossible for the characteristic roots to be purely imaginary. That is, his conjecture is that when the integrand is strictly concave in k and $\dot{k}$ it is always the case that the optimal path has a saddle-point behavior. Kurz, on the other hand, raised the possibility of total instability for the case $\delta=0$. Kurz does not state, however, whether he is discussing the case of a concave integrand or not. In this paper, we prove as Samuelson conjectures, that when $\delta=0$ the optimal path around the steady state has in-

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2) The case of total instability is connected with multiple optimal steady state in which some are saddle-points and some are unstable [2,3].

deed the saddle-point behavior for strictly concave V.

All the theorems are proved also for the discrete time model. In this case, the aim is to maximize $\sum_{t=0}^{\infty} \beta^t V(k_t, k_{t-1})$ where k_t is n-vector of per capita capitals at t, β is the discount factor $(0 < \beta \leq 1)$ and V is a 2n arguments function giving the maximum utility of consumption for known k_t and k_{t-1}. We prove that the characteristic roots come in pairs of λ and $\frac{1}{\beta\lambda}$. In the case $\beta=1$ and V strictly concave it is impossible for any of the roots to be of absolute value 1 and we always have a saddle-point. This is the discrete time counterpart of the theorem stating that in continuous-time models we cannot have purely imaginary roots when $\delta=0$ and V strictly concave.

2. The Continuous and Discrete Models.

In the continuous case we wish to maximize $\int_0^{\infty} e^{-\delta t} V(k,\dot{k})dt$. As shown in [4] and [5] the Taylor's expansion of Euler Equations around the steady state $(\dot{k}=0)$ yields

$$(1) \quad (V_{\dot{k}_i\dot{k}_j})(\ddot{y}) + [(V_{\dot{k}_i k_j})-(V_{\dot{k}_j k_i})-\delta(V_{\dot{k}_i\dot{k}_j})]\,\dot{y} - [(V_{k_i k_j})+\delta V_{\dot{k}_i k_j}](y) = 0$$

where $y = k-k^*$ are n-vectors of deviations from optimal steady state values and where $(V_{\dot{k}_i\dot{k}_j}),(V_{\dot{k}_i k_j}),(V_{k_i\dot{k}_j}),(V_{k_i k_j})$, with $i,j = 1,\ldots,n$, are notations for the $n\times n$ matrices composed of the appropriate partial derivatives evaluated at an optimal steady state. Denote $A = (V_{\dot{k}_i\dot{k}_j})$, $B = (V_{\dot{k}_i k_j})$, $C = (V_{k_i k_j})$, and we observe $B' = (V_{\dot{k}_j k_i})$ (prime denotes the transposed matrix). The characteristic equation of this system is:

(2) $\|A\lambda^2 + (B-B'-\delta A)\lambda - (C+\delta B')\| = 0$.

Under the assumption of strict concavity the matrix $\begin{pmatrix} A & B \\ B' & C \end{pmatrix}$ is negative definite.

For the discrete case, as mentioned, the aim is to maximize $\sum_{t=0}^{\infty} \beta^t V(k_t, k_{t-1})$ where V gives maximal utility for given k_t, k_{t-1}. Taylor expansion of the system of Euler-like difference equations yields:[3)]

(3) $$\beta(V_{k^i_{t-1}k^j_t})(x_{t+2}) + [(V_{k^i_t k^j_t}) + \beta(V_{k^i_{t-1}k^j_{t-1}})](x_{t+1}) + \\ + (V_{k^i_t k^j_{t-1}})(x_t) = 0$$

where x_t are n-vectors of deviations from optimal steady state values, i.e., $x_t = k_t - k^*$.

We denote $(V_{k^i_t k^j_t}) = A$, $(V_{k^i_t k^j_{t-1}}) = B$, $(V_{k^i_{t-1}k^j_{t-1}}) = C$ and observe $V_{k^i_{t-1}k^j_t} = B'$. The characteristic equation of the Euler-like system of difference equations is

(4) $\|\beta B'\lambda^2 + (A+\beta C)\lambda + B\| = 0$.

Characterizations of the Optimal Path

Theorem 1. If λ is a characteristic root of the polynomial equation (2) then $-\lambda+\delta$ is a root as well.

Proof. Let us substitute $-\lambda+\delta$. This yields with trivial calculations

$$A(\delta-\lambda)^2 + (B-B'-\delta A)(\delta-\lambda) - (C+\delta B) = A\lambda^2 + (B-B'+\delta A)(-\lambda) - (C+\delta B')$$

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3) See the analogous case in Samuelson [6,7] with characteristic equation of the same form.

However, we already know that $|A\lambda^2 + (B-B'-\delta A)\lambda - (C+\delta B)'| = 0$ and as transposing does not change the value of the determinant, we find that $-\lambda+\delta$ is also a root. This theorem gives Kurz's conclusions that the only possibilities are either saddle-point or complete instability. We cannot have complete stability since if, say, the real part of λ is negative then the real part of the related root $-\lambda+\delta$ cannot be negative too.

In the following theorem, we show that in the case $\delta=0$ we always have a saddle-point, that is purely imaginary characterixtic roots are impossible.

Theorem 2. The characteristic equation $|A\lambda^2 + (B-B')\lambda - C|$ posesses no purely imaginary root.

Proof. Assume $\lambda = i\beta$ to be a characteristic root. The matrix $A\lambda^2 + (B-B')\lambda - C$ possesses a non-trivial solution $(A\lambda^2 + (B-B')\lambda - C)(x+iy) = 0$ where x and y are n-dimensional real vectors not both zero. Multiplying this equation by $(x-iy)'$ we get

$$(x-iy)'[A\lambda^2 + (B-B')\lambda - C](x+iy) = 0 \ . \tag{5}$$

Let us now compute the following quadratic form (defining $\bar{\lambda}$ to be the conjugate of λ):

$$[\lambda(x-iy)',(x-iy)'] \begin{pmatrix} A & B' \\ B & C \end{pmatrix} \begin{bmatrix} \bar{\lambda}(x+iy) \\ x+iy \end{bmatrix} = \lambda\bar{\lambda}(x-iy)'A(x+iy) + \tag{6}$$

$$+ \bar{\lambda}(x-iy)'B(x+iy) + \lambda(x-iy)'B'(x+iy) + (x-iy)'C(x+iy) \ .$$

As λ is purely imaginary $\bar{\lambda} = -\lambda$ and we obtain for (6)

$$-(x-iy)'A(x+iy)\lambda^2 + (x-iy)'(B'-B)(x+iy)\lambda + (x-iy)'C(x+iy) \ . \tag{7}$$

Thus, we see that:

$$(8)\quad [\lambda(x-iy);x-iy]'\begin{pmatrix}A & B'\\ B & C\end{pmatrix}\begin{bmatrix}\bar{\lambda}(x+iy)\\ x+iy\end{bmatrix} =$$

$$= -(x-iy)'[A\lambda^2 + (B-B')\lambda - C](x+iy) = 0$$

where the equality to zero follows from (5).

However, the matrix $\begin{pmatrix}A & B'\\ B & C\end{pmatrix}$ is a symmetric negative definite matrix and for any negative definite symmetric matrix H, the quadratic for $z'H\bar{z}$, as in (5), is negative for $z \neq 0$. Hence, the left hand side of (8) should be negative, a contradiction. It follows that all the roots of this characteristic equation must have non-vanishing real parts.

Theorem 3. (The discrete case). If λ is a solution of equation (4) then $\frac{1}{\lambda\beta}$ is also a solution.

Proof. By substitution in (4) we obtain:

$$\beta B'\left(\frac{1}{\lambda\beta}\right)^2 + (A+\beta C)\frac{1}{\lambda\beta} + B .$$

Multiplying all the elements by $\beta\lambda^2$ the determinant of this matrix is multiplied by $\beta^n\lambda^{2n}$ and we get $B' + (A+\beta C)\lambda + B\beta\lambda^2$. Transposing, we finally have $\beta B'\lambda^2 + (A+\beta C)\lambda + B$. However, we already know that $\|\beta B'\lambda^2 + (A+\beta C)\lambda + B\| = 0$. Thus, $\frac{1}{\lambda\beta}$ is also a root.*

We find then that it is impossible for both roots to be within the unit circle. In the case $1 < |\lambda| < \frac{1}{\beta}$ both roots will be outside the unit circle and we shall obtain the unstable case. In the case of no discounting, $\beta=1$, the only conceivable case of non-saddle-point is the case of $|\lambda| = 1$. However, the following

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* All this assumes that $\lambda \neq 0$ or that $\|B\| \neq 0$; we neglect this singular case.

theorem shows that in the strictly concave case this is impossible.

Theorem 4. The equation $\|B'\lambda^2 + (A+C)\lambda + B\| = 0$ possesses no roots on the unit circle.

Assume that this equation possesses a root on the unit circle, i.e., λ is a root with $\lambda\cdot\bar{\lambda} = 1$.

As $B'\lambda^2 + (A+C)\lambda + B$ is a singular matrix, there is a non-trivial solution to the system.

$$[B'\lambda^2 + (A+C)\lambda + B](x+iy) = 0 .$$

Multiplying on the left side with $\bar{\lambda}(x-iy)'$ and using $\lambda\bar{\lambda} = 1$ we find

$$(9) \quad (x-iy)'[B'\lambda + (A+C) + B\bar{\lambda}](x+iy) = 0 .$$

Let us look at the following quadratic form

$$(10) \quad [\bar{\lambda}(x-iy)', (x-iy)'] \begin{pmatrix} A & B \\ B' & C \end{pmatrix} \begin{bmatrix} \lambda(x+iy) \\ (x+iy) \end{bmatrix} = (x-iy)'(A+C)(x+iy) +$$

$$(x-iy)'B(x+iy)\bar{\lambda} + (x-iy)'B'(x+iy)\lambda =$$

$$(x-iy)'[B'\lambda + (A+C) + B\bar{\lambda}](x+iy) = 0 .$$

However, as $\begin{pmatrix} A & B \\ B' & C \end{pmatrix}$ is negative definite symmetric matrix multiplied on both sides by conjugate vectors it should be negative, a contradiction. Thus, also for this case, we find that $\beta=1$ is always a case of saddle-point and again, complete instability is impossible in the case where V is strictly concave.

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